COMPLETELY POSITIVE MAPS AND EXTREMAL K-SET

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ABSTRACT. We introduce a new set $K_e(A, B)$ where A is a commutative C^*-algebra, B is a C^*-algebra. It contains $KK(A, B)$. When $A = S$ where S is the suspension, we show there is a nice interpretation for $K_e(A, B)$.

1. INTRODUCTION

Since Kasparov invented his celebrated bivariant KK-functor in 1981 [Kas2], KK-theory was studied by several mathematicians-Joachim Cuntz, Goerge Skandalis, Nigel Higson, Claude Schochet, Jonathan Rosenberg during 1981-90. The power and utility have been fully demonstrated by its applications to geomtry, topology and recently Elliott's Classification program since then.

The close connection between KK-theory and K-theory is one of main features of KK-theory which is also the basic fact for the Universal Coefficient Theorem (shortly UCT). This paper is a try to pursue this point of view further along the introduction of Brown and Pedersen 's K_e and E_∞[BrPed]. With the account of the Cuntz's description of KK-group, our goal is to find the appropriate counterpart for $K_e(B)$ and $E_\infty(B)$ where B is a C^*-algebra. It turns out that a slight variation of Cuntz's picture also gives us the right candidate (See §4 and §5).

The plan of this papaer is as follows. After dealing with some preliminaries on completely positive mapping in §2, we review definitions of E_∞ and K_e and summarize the basic facts which is necessary for our purpose in §3.(For our goal, stability of these groups are essential.) In §4, we define $[\mathcal{E}(A, B)]$ and $KK_e(A, B)$ for A which is in a small category of commutative C^*-algebras. In contrast with the definition KK-group, our definition shows a generality of a variable in KK-theory

comes from the restriction about morphisms. Finally, in §5, we establish the connection between $[\mathcal{E}(A,B)]$, $KK_e(A,B)$ and $E_\infty(B)$, $K_e(B)$ respectively which generalizes the connection $KK^1(A,B)$ and $K_1(B)$.

2. COMPLETELY POSITIVE MAP BETWEEN C^*-ALGEBRAS

In this section, we show some results about completely positive map of C^*-algebras which will be useful later.

For the definition and basic facts of completely positive map of C^*-algebra, we recommend you Paulsen's Book [P] as a general reference.

Theorem 2.1 (Stinespring's Dilation Theorem). *Let A be a (not necessarily unital) C^*-algebra and $\mathcal{B}(\mathbb{H})$ be a space of the bounded linear operators on Hilbert space $\mathbb{H}$. If $\phi : A \to \mathcal{B}(\mathbb{H})$ is completely positive map, then there exist a Hilbert space $\mathbb{K}$, a nondegenerate *-representation $\pi : A \to \mathcal{B}(\mathbb{K})$, and a bounded operator $V : \mathbb{H} \to \mathbb{K}$ such that*

$$\phi(a) = V^*\pi(a)V \text{ for all } a \in A$$

proof. A general version of Stinespring Dilation theorem was suggested and proven by Kasparov [Kas1] in Hilbert C^*-module setting. Most elaborate proof is found in P48-52 [Lan]

Lemma 2.2. *Let A and B be C^*-algebras. If $\phi : A \to B$ is completely positive, then ϕ is completely bounded and $\|\phi\|_{cb} = \sup_n \|\phi_n\| = \|\phi\|$. Consequently, ϕ is isometric(contractive), ϕ is completely isometric(contractive).*

proof. By the Gelfand-Naimark theorem, we can assume $\phi : A \to \mathcal{B}(\mathbb{H})$ for some Hilbert space $\mathbb{H}$. then by the above theorem, there exist a Hilbert space $\mathbb{K}$, a *-representation $\pi : A \to \mathcal{B}(\mathbb{K})$, and a bounded operator $V : \mathbb{H} \to \mathbb{K}$ such that $\phi(a) = V^*\pi(a)V$. Since

$$\|\phi_n((a_{i,j}))\| \leq \|V^*\|\|(a_{i,j})\|\|V\| \text{ for each } n,$$

$\|\phi\|_{cb} \leq \|V\|^2$, thus, ϕ is completely bounded. But if (e_i) is an approximate unit for A, using $\phi(a^*e_i)\phi(e_ia) \leq \|\phi(e_i^2)\|\phi(a^*a)$, we can deduce $\|\phi\| = \sup_i \|\phi(e_i)\|$. Hence $\|\phi\| = \|V\|^2$. Finally, since $e_i^{(n)} = e_i \otimes 1 \in A \otimes M_n(\mathbb{C}) \cong M_n(A)$ is an approximate unit for $M_n(A)$ and $\|\phi_n(e_i^{(n)})\| = \|\phi(e_i)\|$, using similar inequality for ϕ_n, we get $\|\phi\| = \sup_i \|\phi(e_i)\| = \sup_i \|\phi_n(e_i^{(n)})\| \leq \|\phi_n\|$ for each n. So we complete the proof.

Now we are ready to prove the following proposition which will be basic tool through this paper. We let $\tilde{A}$ be a unitization of A.

Proposition 2.3. *Let A be non-unital, B be an unital C^*-algebra and $\phi : A \to B$ be completely positive and contractive. Then there is a unital map $\tilde{\phi} : \tilde{A} \to B$ which is also completely positive and extends ϕ. Moreover, such a map is unique.*

proof. The idea of this proof is borrowed from Huaxin Lin. We define $\tilde{\phi}(a + \lambda) = \phi(a) + \lambda 1$. Clearly, $\tilde{\phi}$ is extends ϕ and unital. To show $\tilde{\phi}$ is completely positive, we must show $\tilde{\phi}_n is positive$. Note that any element $b \in M_n(\tilde{A})$ can be written down as $a + s$ where $a \in M_n(A)$ and $s \in M_n(\mathbb{C}1)$.

Suppose b is positive. By considering natural quotient map

$$M_n(\tilde{A}) \to \frac{M_n(\tilde{A})}{M_n(A)}$$

s is also positive. For any $\epsilon > 0$, if we set $s_\epsilon = s + \epsilon 1$, then s_ϵ is invertible. Since $a + s_\epsilon$ is positive, we have

$$-(s_\epsilon)^{-1/2} a (s_\epsilon)^{-1/2} \leq 1_{M_n(\tilde{A})}$$

Let (e_λ) be an approximate identity for $M_n(A)$. Thus

$$-e_\lambda (s_\epsilon)^{-1/2} a (s_\epsilon)^{-1/2} e_\lambda \leq (e_\lambda)^2.$$

Since $(s_\epsilon)^{-1/2}$ is also scalar matrix, $(s_\epsilon)^{-1/2} a (s_\epsilon)^{-1/2} \in M_n(A)$ and

$$e_\lambda (s_\epsilon)^{-1/2} a (s_\epsilon)^{-1/2} e_\lambda \to (s_\epsilon)^{-1/2} a (s_\epsilon)^{-1/2}.$$

Since ϕ is completely contractive by lemma 2.2

$$\phi_n(-e_\lambda (s_\epsilon)^{-1/2} a (s_\epsilon)^{-1/2} e_\lambda) \leq \phi_n((e_\lambda)^2) \leq 1_{M_n(B)}$$

Observe that $\phi_n(sa) = s\phi_n(a)$ and $\phi_n(as) = \phi_n(a)s$ hold for any scalar matrix s. From this observation, we obtain

$$-s_\epsilon^{-1/2} \phi_n(a) s_\epsilon^{-1/2} \leq 1_{M_n(B)}$$

Consequently,

$$\phi_n(a) + S_\epsilon \geq 0$$

Thus

$$\tilde{\phi}(a + s) = \phi_n(a) + s \geq -\epsilon 1_{M_n(B)}$$

Now let $\epsilon \to 0$. we get

$$\tilde{\phi}(a + s) \geq 0.$$

So we have shown $\tilde{\phi}_n$ is positive. Uniqueness part is obvious.

Lemma 2.4. *Let A be a commutative C^*-algebra and B a C^*-algebra. If $\phi : A \mapsto B$ is positive, then ϕ is completely positive.*

proof. (Unital case) If A is a unital commutative C^*algebra, then we can assume that $A = C(X)$ where X is a compact Hausdorff space. Let $\epsilon > 0$ be given and $P(x)$ be positive in $\mathbb{M}_n(C(X))$. We must prove $\phi_n(P)$ is positive. Using a partition of unity $\{u_l(x)\}$ subordinate to a covering $\{O_l\}$ such that $\|P(x) - P(x_l)\| < \epsilon$ for $x \in O_l$ and positive matrices $P(x_l) = P_l = (P_{i,j})$, we have

$$\|P(x) - \sum_l u_l(x)P_l\| < \epsilon$$

But $\phi_n(u_lP_l) = (\phi(u_l)p_{i,j})$ which is postive in $\mathbb{M}_n(C(X))$. Since $M_n(B)^+$ is closed set, $\phi_n(P)$ is positive.

(Non-unital case) If A is non-unital, we extend ϕ to $\tilde{A}$ the unitization of A. Define $\tilde{\phi} : \tilde{A} \to \tilde{B}$ by $\tilde{\phi}(a + \lambda 1) = \phi(a) + \lambda\|\phi\|1$. Note that if $a + \lambda 1$ is positive, then $\lambda \geq 0$. From this, $\tilde{\phi}$ is also positive. Hence $\tilde{\phi}$ is completely positive as we have seen above. Consequently, Restriction of $\tilde{\phi}$ to A which is ϕ is also completely positive.

Lemma 2.5. *Let B be a unital C^*-algebra and A be a C^*- algebra(not necessarily unital) $\phi : A \to B$ be a completely positive contractive map. then*

(1) $\{a \in A|\, \phi(a)\phi(a^*) = \phi(aa^*)\} = \{a \in A|\, \phi(a)\phi(b) = \phi(ab)\,for\,all\,b \in A\}$ *is a subalgebra of A and ϕ is a homomorphism when it is restricted to this set.*

(2) $\{a \in A|\, \phi(a^*)\phi(a) = \phi(a^*a)\} = \{a \in A|\, \phi(b)\phi(a) = \phi(ba)\,for\,all\,b \in A\}$ *is a subalgebra of A and ϕ is a homomorphism when it is restricted to this set.*

(3) $\{a \in A|\, \phi(a)\phi(a^*) = \phi(aa^*)\&\phi(a^*a) = \phi(a^*)\phi(a)\}$ $= \{a \in A|\, \phi(a)\phi(b) = \phi(ab)\&\phi(ba) = \phi(b)\phi(a)\,for\,all\,b \in A\}$ *is a subalgebra of A and ϕ is a *-homomorphism when it is restricted to this set.*

proof. we prove (1). The proofs of (2) and (3) are similar. We may assume that $\phi : A \mapsto \mathcal{B}(\mathbb{H})$ where $\mathbb{H}$ is a Hilbert space. By the theorem, there is a Hilbert space $\mathbb{K}$ containing $\mathbb{H}$, $V \in \mathcal{B}(\mathbb{K})$ with $\|V\| \leq 1$ and $\pi : A \mapsto \mathcal{B}(\mathbb{K})$ is a *-representation of A such that $\phi(a) = V^*\pi(a)V$ for all a.

Let a belong to the set on the left. then since $\phi(a)\phi(a^*) = \phi(aa^*)$ holds, we have $V^*\pi(a)(1 - VV^*)\pi(a)V = 0$. Note that $\|V\| \leq 1$. This implies that $1 - VV^*$ is positive. Hence

$$V^*\pi(a)(1 - VV^*)^{1/2}(1 - VV^*)^{1/2}\pi(a^*)V = 0$$

Consequently, $V^*\pi(a)(1 - VV^*)^{1/2} = 0$. Then

$$
\begin{aligned}
V^*\pi(a)(1 - VV^*)^{1/2}(1 - VV^*)^{1/2}\pi(b)V &= 0 \text{ for all } b \in A \\
V^*\pi(a)(1 - VV^*)\pi(b)V &= 0 \text{ for all } b \in A \\
\therefore V^*\pi(a)VV^*\pi(b)V &= V^*\pi(ab)V
\end{aligned}
$$

So we have shown that a is the element of the set on the right.

Remark 2.6. *(i) We call the set in (1)* left multiplicative domain *for ϕ, the set in (2)* right multiplicative domain *for ϕ and the set in (3)* multiplicative domain *for ϕ.*
(ii) There is a more general virsion of above lemma. See the theorem 3.18 in [P].

Corollary 2.7. *Let $\phi : S \to \mathcal{M}(B \otimes \mathcal{K})$ be completely positive contractive map. If $\phi(f) + 1$ is a unitary, then ϕ is *-homomorphism.*

proof If $\phi(f)+1$ is a unitary, then $\phi(f)\phi(f^*) = \phi(ff^*)\&\phi(f^*)\phi(f) = \phi(f^*f)$ hold. Note that f is a generator of S. Hence $\{a \in S | \phi(a)\phi(a^*) = \phi(aa^*)\&\phi(a^*a) = \phi(a^*)\phi(a)\}$ is nonempty and is S itself. $\therefore \phi$ is a *-homomorphism by lemma2.5.

Proposition 2.8. *Let A be a C^*-algebra(not necessarily unital), B be a unital C^*-algebra. Let $\phi : A \to B$ be a positive map. Assume I_+, I_- are centrally orthogonal ideals in B. Let π^+, π^- be the natural quotient maps from B onto $B/I_+, B/I_-$ respectively. If*

$$
\begin{aligned}
\phi^+ &= \pi^+ \circ \phi : A \to B \to B/I_+ \\
\phi^- &= \pi^- \circ \phi : A \to B \to B/I_-
\end{aligned}
$$

are completely positive contractive, then $\phi : A \to B$ is completely positive and contractive.

proof. Consider the *-homomorphism $\pi^+ \oplus \pi^- : B \to B/I_+ \oplus B/I_-$. Then this map is injective since $I^+ \cap I^- = 0$. Since the map $\phi^+ \oplus \phi^- : A \to B/I_+ \oplus B/I_-$ is completely positive by the assumtion, ϕ is completely positive. In fact, $B \cong \{x \oplus y \in B/I_+ \oplus B/I_- \mid$ Image of x = Image of y in $B/I^+ + I^-\}$. Hence $\|\phi(a)\| = \max\{\|\phi^+(a)\|, \|\phi^-(a)\|\} \leq \|a\|$. So we have shown ϕ is also contractive.

3. STABLE EXTREMAL CLASS AND $K_e(-)$

In this section, we summarize the definitions of $E_\infty(-)$ and $K_e(-)$ and basic results from [BrPed]. Throughout this section A will denote a unital C^*-algebra and $\mathcal{E}(A)$ the set of extreme points in the unit ball A_1 of A, that is, the patial isometries v such that $(1 - v^*v)A(1 - vv^*) = 0$. The centrally orthogonal projections $p_+ = 1 - v^*v$ and $p_- = 1 - vv^*$

will be referred to as the *defect projections* of v, and the two orthogonal closed ideals I_+ and I_- generated by these projections will be known as the *defect ideals* of v.

Theorem 3.1. *Given $\epsilon > 0$ there is a $\delta > 0$, such that for any pair v, w in $\mathcal{E}(A)$ with $\|v - w\| \leq \delta$, there are unitaries u_1 and U_2 in A with $v = u_1 w u_2$ and $\|1 - u_i\| \leq \epsilon$ for $i = 1, 2$.*

proof This is the theorem 2.1 in [BrPed].

Corollary 3.2. *Two elements v and w in $\mathcal{E}(A)$ are homotopic if and only if $w = u_1 w u_2$ for some unitaries u_1, u_2 in $\mathcal{U}_0(A)$, the connected component of the unitary group $\mathcal{U}(A)$ containing $\mathbf{1}$.*

proof. See the corollary 2.3. in [BrPed].

Proposition 3.3. *Let v and w be extremal partial isometries in A, and consider the defect projections $p_+ = 1 - v^*v, p_- = 1 - vv^*$, and $q_+ = 1 - w^*w, q_- = 1 - ww^*$; and the corresponding defect ideals I_+, I_- and J_+, J_-. Then the following are equivalent.*

(1) $p_+ A q_- = p_- A q_+ = \{0\}$,
(2) $I_+ \cap J_- = I_- \cap J_+ = \{0\}$,
(3) $vw \in \mathcal{E}(A)$ and $wv \in \mathcal{E}(A)$,
(4) $\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in \mathcal{E}(M_2(A))$

proof. This is the proposition 2.5 in [BrPed].

When the conditions in proposition3.3 are satisfied, we say that v and w are *composable*. We see the unitary elements are composable to all other elements in $\mathcal{E}(A)$, and also that they are the only elements composable their adjoints.

Corollary 3.4. *Suppose v and w are composable. If v_1 is homotopic to v and w_1 is homotopic to w, then v_1 and w_1 is also composable.*

proof It is easy to check that defect ideals of v_1 and w_1 are exactly the defect ideals of v and w respectively. From this, the conclusion is straightforward.

Proposition 3.5. *When v and w are composable, two defect ideals for vw are precisely $I_+ + I_-$ and $I_- + J_-$.*

proof Note that $P_+ = 1 - (vw)^*vw = q_+ + w^*p_+w$. Defect ideal which is generated by $P_+ = 1 - (vw)^*vw$ is contained in $I_+ + I_-$. Conversely, from $wP_+w^* = p_+ and P_+q_+ = q_+$, $I_+ + I_-$is also contained the ideal generated by P_+. The other case is proved similarly.

Definition 3.6. *Let A be a unital C^*-algebra and for each n, consider the set $[\mathcal{E}(\mathbb{M}_n(A))]$ of homotopy classes of extreme partial isometires in the algebra $\mathbb{M}_n(A)$. The embeddings*

$$\iota_{mn} : v \hookrightarrow \begin{pmatrix} v & 0 \\ 0 & 1_{m-n} \end{pmatrix}$$

of $\mathcal{E}(\mathbb{M}_n(A))$ into $\mathcal{E}(\mathbb{M}_m(A))$, for $1 \leq n < m$, evidently respect homotopy, we define $[\mathcal{E}_\infty(A)] = \varinjlim[\mathcal{E}(\mathbb{M}_n(A))]$ in complete analogy with the definition of $K_1(A)$. We shall refer this set as the set of stable extremal classes *for A and we shall denote by $[v]$ its homotopy class in $[\mathcal{E}_\infty(A)]$.*

Proposition 3.7. *Let A be a C^*-algebra. Then*

$$[\mathcal{E}_\infty(A)] \cong [\mathcal{E}((A \otimes \mathcal{K})^\sim)]$$

proof. Note that ι_{mn} induce the isomorphism betweem $[\mathcal{E}_\infty(\mathbb{M}_n(A))]$ and $[\mathcal{E}_\infty(\mathbb{M}_m(A))]$ for $1 \leq n < m$. In fact, any 'corner' embeddings $id \otimes e_{ii}$ are all homotopic.
If $A = \varinjlim A_i$ is an s.e.p.p. inductive limit of C^*-algebras(i.e. every connecting morphism $\iota_{i,j}$ is stably extremal preserving maps), then $[\mathcal{E}_\infty(A)] = \varinjlim[\mathcal{E}_\infty(A_i)]$ if each $\iota_{i,j}$ is injective.(See P218 in [BrPed]).
Then by combining these two observations, we see that $[\mathcal{E}_\infty(A \otimes \mathcal{K})] = [\mathcal{E}_\infty(A)]$.
Since $[\mathcal{E}(M_n(A \otimes \mathcal{K}))] \cong [\mathcal{E}(A \otimes \mathcal{K} \otimes \mathbb{M}_n)] \cong [\mathcal{E}(A \otimes \mathcal{K})]$, $[\mathcal{E}_\infty(A \otimes \mathcal{K})] \cong [\mathcal{E}(A \otimes \mathcal{K})] = [\mathcal{E}((A \otimes \mathcal{K})^\sim)]$

Proposition 3.8. *Let A be a C^*-algebra. If $id \otimes e_{11} : A \mapsto A \otimes \mathcal{K}$ is the corner embedding, then $K_e(id \otimes e_{11})$ is an isomorphism between $K_e(A)$ and $K_e(A \otimes \mathcal{K})$.*

proof. The argument is almost identical to proposition 3.7. See page 219 4.12 (iv) of [BrPed].

4. THE SET $[\mathcal{E}_\infty(-,-)]$ AND $KK_e(-,-)$

Based upon Brown and Pedersen's work [BrPed], we define more general functor $[\mathcal{E}_\infty(A, B)]$ where A is a C^*-algebra generated by $u - 1$ where u is an unitary on a Hilbert space and B is σ-unital C^*-algebra.

Definition 4.1. *Let $\mathcal{E}(A, B)$ be the set of the pairs (ϕ_+, ϕ_-) s.t.*
  (1) *$\phi_+ : A \to \mathcal{M}(B \otimes \mathcal{K})$ *-homomorphism, $\phi_- : A \to \mathcal{M}(B \otimes \mathcal{K})$ completely positive contractive s.t.*
    (a) *$\tilde{\phi}_-(u)$ is extremal partial isometry for the unitary element $u \in \tilde{A}$*

(b) $\tilde{\phi}_-(u^n) = (\tilde{\phi}_-(u))^n$ $\tilde{\phi}_-((u^*)^n) = (\tilde{\phi}_-(u^*))^n$ *for the unitary element* $u \in \tilde{A}$

(2) $\tilde{\phi}_+(u) - \tilde{\phi}_-(u) \in B \otimes \mathcal{K}$ *thus* $(\tilde{\phi}_+(u))^* \tilde{\phi}_-(u) \in 1 + B \otimes \mathcal{K}$

We call such a pair (ϕ_+, ϕ_-) as *a generalized extremal cycle* and two generalized extremal cycles $(\phi_+, \phi_-), (\psi_+, \psi_-)$ are *homotopic* when there is a generalized extremal cycle from A to $\mathcal{M}([0,1]B \otimes \mathcal{K})$ i.e. $(\lambda_+^t, \lambda_-^t) \in \mathcal{E}(A, B), t \in [0, 1]$ s.t.

(1) the maps $t \to \tilde{\lambda}_+^t(u)$ and $t \to \tilde{\lambda}_-^t(u)$ from $[0, 1]$ to $\mathcal{M}(B \otimes \mathcal{K})$ are strictly continuous.

(2) $\tilde{\lambda}_+^t(u) - \tilde{\lambda}_-^t(u) \in B \otimes \mathcal{K}$ for each t. and the map $t \to \tilde{\lambda}_+^t(u) - \tilde{\lambda}_-^t(u)$ from $[0, 1]$ to $B \otimes \mathcal{K}$ is norm continuous and thus the map $t \to (\tilde{\lambda}_+^t(u))^* \tilde{\lambda}_-^t(u)$ from $[0, 1]$ to $1 + B \otimes \mathcal{K}$ is norm continuous

We write $(\phi_+, \phi_-) \sim (\psi_+, \psi_-)$ in this case.

Definition 4.2. *We let* $[\mathcal{E}_\infty(A, B)] \overset{def}{=} \mathcal{E}(A, B)/ \sim$ *denote the homotopy classes of generalized extremal cycles. The homotopy classes in* $E(A, B)$ *represented by* $(\phi_+, \phi_-) \in \mathcal{E}(A, B)$ *is denoted by* $[\phi_+, \phi_-]$.

We shall refer to $[\mathcal{E}_\infty(A, B)]$ as the set of *stable extremal classes* of A and B.

Proposition 4.3. $KK(A, B) \subseteq [\mathcal{E}_\infty(A, B)]$.

proof. Since $KK(A, B) = \{[\phi_+, \phi_-] \mid \phi_+(a) - \phi_-(a) \in B \otimes \mathcal{K}$ for each $a \in A\}$, where $\phi_+, \phi_- \in \text{Hom}(A, \mathcal{M}(B \otimes \mathcal{K}))$(See Chapter4 in [JenThom]), their unital extensions $\tilde{\phi}_+, \tilde{\phi}_-$ are *-homomorphisms. In particular, $\tilde{\phi}_-$ satisfies the conditions (a)and (b) in definition4.1. Since unitaries are composable to any extremal partial isometry, $\tilde{\phi}_+(u)^* \tilde{\phi}_-(u), \tilde{\phi}_+(u)\tilde{\phi}_-(u)^*$ is of the form $1 + B \otimes \mathcal{K}$ because $\tilde{\phi}_+(u) - \tilde{\phi}_-(u) \in B \otimes \mathcal{K}$. i.e. (ϕ_+, ϕ_-) is in $\mathcal{E}(A, B)$. We can apply the same argument to a homotopy between two KK-cycles so that it is well-defined.

We can define a partial addition for manageable extremal classes.

Definition 4.4. *We say two generalized extremal cycles* (ϕ_+, ϕ_-) *and* (ψ_+, ψ_-) *are* composable *if* $\tilde{\phi}_-(u)$ *and* $\tilde{\psi}_-(u)$ *are composable in* $\mathcal{E}(\mathcal{M}(B \otimes \mathcal{K}))$ *for* u.

Suppose two generalized extremal cycles (ϕ_+, ϕ_-) and (ψ_+, ψ_-) are *composable*. Then we can check the following facts.

(1) $\Theta_B \circ \begin{bmatrix} \tilde{\phi}_+ & 0 \\ 0 & \tilde{\psi}_+ \end{bmatrix}$ is *-homomorphism , $\Theta_B \circ \begin{bmatrix} \tilde{\phi}_- & 0 \\ 0 & \tilde{\psi}_- \end{bmatrix}$ is also completely positive map which satisfies the conditions $(a), (b)$ of definition 4.1.

(2) $\left(\Theta_B \circ \begin{bmatrix} \tilde{\phi}_+ & 0 \\ 0 & \tilde{\psi}_+ \end{bmatrix} \right)(u) - \left(\Theta_B \circ \begin{bmatrix} \tilde{\phi}_- & 0 \\ 0 & \tilde{\psi}_- \end{bmatrix} \right)(u) \in B \otimes \mathcal{K}$

$\left(\Theta_B \circ \begin{bmatrix} \tilde{\phi}_+ & 0 \\ 0 & \tilde{\psi}_+ \end{bmatrix} \right)^*(u) \left(\Theta_B \circ \begin{bmatrix} \tilde{\phi}_-(u) & 0 \\ 0 & \tilde{\psi}_-(u) \end{bmatrix} \right)(u)$

are of the form $1 + B \otimes \mathcal{K}$ for u.

where $\Theta_B : M_2(\mathcal{M}(B \otimes \mathcal{K})) \to \mathcal{M}(B \otimes \mathcal{K})$ is an inner *-isomorphism. Now we can define an addition between two *composable* generalized extremal classes by

$$[\phi_+, \phi_-] + [\psi_+, \psi_-] = \left[\Theta_B \circ \begin{bmatrix} \phi_+ & 0 \\ 0 & \psi_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \phi_- & 0 \\ 0 & \psi_- \end{bmatrix} \right]$$

It is easy to check each element in $KK(A, B)$ is *composable* to an element in $E(A, B)$. This implies our definition of *composability* follows the same spirit of the definition of *composability* in [BrPed]. Next result consolidates our definition of *composability* is exactly analogous to and extended notion of the definition of *composability* of extremal partial isometries.

Proposition 4.5. *The addition is associative whenever possible.*

proof If α, β, and γ are elements in $E(A, B)$ such that α is composable with β and $\alpha + \beta$ is composable with γ, then β is composable with γ and α is composable with $\beta + \gamma$. This follows by observing that defect ideal for a sum of elements is the sum of the defect ideals for the summands. Then using rotational homotopies finally we have $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ (See Lemma 1.3.12 in [JenThom]).

We summarize our observations in the following theorem.

Theorem 4.6. *For a non-unital C^*-algebra A which is generated by $u - 1$ where u is an unitary on a Hilbert space and σ-unital C^*-algebra B the set of extremal classes of A and B $[\mathcal{E}_\infty(A, B)]$ is a set with a partially defined addition between composable elements. There is a natural embedding $KK(A, B) \subset [\mathcal{E}_\infty(A, B)]$, and addtion in $[\mathcal{E}_\infty(A, B)]$ extends the addition in $KK(A, B)$.*

As Brown and Pedersen have defined a coarser equivalence relation than homotopy[BrPed] , we introduce a coarser equivalence relation than homotopy to $\mathcal{E}(A, B)$.

Definition 4.7. *For any two elements $\alpha = [\phi_+, \phi_-]$ and $\beta = [\psi_+, \psi_-]$ in $E(A, B)$, we define $\alpha \approx \beta$ in $E(A, B)$ if there is an element $(\tau_+, \tau_-) \in \mathcal{E}(A, B)$ s.t.*

(1) *$\tilde{\tau}_+(u)$ has smaller defects than $\tilde{\phi}_+(u), \tilde{\psi}_+(u)$.*
(2) *$\tilde{\tau}_-(u)$ has smaller defects than $\tilde{\phi}_-(u), \tilde{\psi}_-(u)$.*
(3) *$[\phi_+, \phi_-] + [\tau_+, \tau_-] = [\psi_+, \psi_-] + [\tau_+, \tau_-]$ or,*

$$\left(\Theta_B \circ \begin{bmatrix} \phi_+ & 0 \\ 0 & \tau_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \phi_- & 0 \\ 0 & \tau_- \end{bmatrix} \right) \sim \left(\Theta_B \circ \begin{bmatrix} \psi_+ & 0 \\ 0 & \tau_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \psi_- & 0 \\ 0 & \tau_- \end{bmatrix} \right)$$

Evidently, we may assume that α and β had the same defect ideals. i.e. $\tilde{\phi}_+(u), \tilde{\psi}_+(u)$ had the same defect ideals and $\tilde{\phi}_-(u), \tilde{\psi}_-(u)$ had the same ideals for each $u \in \mathcal{E}(A)$.

To verify this is an equivalence relation, we only prove transitivity part.
Now we let $\alpha = [\phi_+^1, \phi_-^1] \approx \beta = [\phi_+^2, \phi_-^2]$ and $\beta = [\phi_+^2, \phi_-^2] \approx \gamma = [\phi_+^3, \phi_-^3]$. There is (μ_+, μ_-) and (ν_+, ν_-) in $\mathcal{E}(A)$ s.t.
(1a)
$$\left(\Theta_B \circ \begin{bmatrix} \phi_+^1 & 0 \\ 0 & \mu_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \phi_-^1 & 0 \\ 0 & \mu_- \end{bmatrix} \right) \sim \left(\Theta_B \circ \begin{bmatrix} \phi_+^2 & 0 \\ 0 & \mu_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \phi_-^2 & 0 \\ 0 & \mu_- \end{bmatrix} \right)$$

and
(1b)
$$\left(\Theta_B \circ \begin{bmatrix} \phi_+^2 & 0 \\ 0 & \nu_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \phi_-^2 & 0 \\ 0 & \nu_- \end{bmatrix} \right) \sim \left(\Theta_B \circ \begin{bmatrix} \phi_+^3 & 0 \\ 0 & \nu_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \phi_-^3 & 0 \\ 0 & \nu_- \end{bmatrix} \right)$$

Note that (μ_+, μ_-) and (ν_+, ν_-) are composable , $\Theta_B \circ \begin{bmatrix} \mu_+(u) & 0 \\ 0 & \nu_+(u) \end{bmatrix}$ has smaller defects than $\phi_+^1(u)$ and $\Theta_B \circ \begin{bmatrix} \mu_-(u) & 0 \\ 0 & \nu_-(u) \end{bmatrix}$ has smaller defects than $\phi_-^1(u)$.

Then

$$\begin{aligned} \alpha + [\mu_+, \mu_-] + [\nu_+, \nu_-] &= \beta + [\mu_+, \mu_-] + [\nu_+, \nu_-] \text{ by (1a)} \\ &= \beta + [\nu_+, \nu_-] + [\mu_+, \mu_-] \\ &= \gamma + [\nu_+, \nu_-] + [\mu_+, \mu_-] \text{ by (1b)} \\ &= \gamma + [\mu_+, \mu_-] + [\nu_+, \nu_-] \end{aligned}$$

We define
$$KK_e(A, B) = [\mathcal{E}_\infty(A, B)]/ \approx$$

and we shall refer to $KK_e(A, B)$ as the *extremal KK-set* of A and B. If α and β are elements in $KK(A, B)$ and $\alpha \approx \beta$, then only choice of $[\tau_+, \tau_-]$ with smaller defect ideals is another element in $KK(A, B)$, whence $\alpha = \beta$. We therefore have a natural embedding of $KK(A, B)$ into $KK_e(A, B)$. The natural class map κ_e from $[\mathcal{E}_\infty(A, B)]$ onto $KK_e(A, B)$ respects composability and addition.(Here composable classes in $KK_e(A, B)$ meaning that one, hence any, pair of representatives in $[\mathcal{E}_\infty(A, B)]$ are composable.)

We summarize our observations in the following theorem.

Theorem 4.8. *For a non-unital C^*-algebra A which is generated by $u - 1$ where u is an unitary on a Hilbert space and σ-unital C^*-algebra B the extremal KK-set of A and B is the set with a partially defined addition between composable elements. There is a natural embedding $KK(A, B) \subset KK_e(A, B)$, and the addition in $KK_e(A, B)$ extends the addition $KK(A, B)$. There is a natural map $\kappa_e : [\mathcal{E}_\infty(A, B)] \rightarrow KK_e(A, B)$ which is surjective and restricts to an isomorphism on $KK(A, B)$.*

5. Application: special case $[\mathcal{E}_\infty(S, -)]$

We begin this section by observing the following proposition. Let $z(t) = e^{2\pi t i} \in C(\mathbb{T})$, $f(t) = e^{2\pi t i} - 1 \in S$.

Proposition 5.1. *Let $\phi : S \rightarrow \mathcal{M}(B \otimes \mathcal{K})$ be a completely positive contractive map and $\tilde{\phi} : C(\mathbb{T}) \rightarrow \mathcal{M}(B \otimes \mathcal{K})$ is the unital extension of ϕ. Then the following are equivalent.*

(1) $\phi(f\bar{f}) - \phi(f)\phi(\bar{f})\mathcal{M}(B \otimes \mathcal{K})\phi(\bar{f}f) - \phi(\bar{f})\phi(f) = 0$
(2) $\phi(f) + 1$ *is a extremal partial isometry.*
(3) $\tilde{\phi}(z)$ *is a extremal partial isometry.*

proof. $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftarrow (4)$ is easy.
For $(1) \Leftarrow (2)$, let $\phi(f) + 1 = V$. Note that $f\bar{f} + f + \bar{f} = 0$. Then it is easy to check $1 - VV^* = \phi(f\bar{f}) - \phi(f)\phi(\bar{f})$. Similarly, $1 - V^*V = \phi(\bar{f}f) - \phi(\bar{f})\phi(f)$.

Proposition 5.2. *Given a completely positive contractive map $\phi : C(\mathbb{T}) \rightarrow B$ a C^*- algebra, if $\phi(z)$ is an extremal partial isometry and $\phi(e^{ih}) = e^{i\phi(h)}$ for any self-adjoint h in $C(\mathbb{T})$, then $\tilde{\phi}$ is an extremal preserving map.*

proof. we assume $\tilde{\phi}(z)$ is an extremal partial isometry, say v. Consider $\phi^+ = \pi^+ \circ \phi : C(\mathbb{T}) \rightarrow \mathcal{M}(B \otimes \mathcal{K}) \rightarrow \mathcal{M}(B \otimes \mathcal{K})/I_+$ as we

have done in proposition 2.8. Note that *the left multiplicative domain* for $\tilde{\phi}_+$ contains z. Hence *the left multiplicative domain* contains z^n for $n \geq 0$. $C(\mathbb{T})$ i.e. $\tilde{\phi}_+$ is homomorphism. Since every extremal partial isometry in $C(\mathbb{T})$ has the form $e^{ih}z^n$ where h is self-adjoint element in $C(\mathbb{T})$, to finish the proof, it is enough to show that $w = \tilde{\phi}(e^{ih}z^n)$ is extremal partial isometry. Since $\tilde{\phi}_+$ is homomorphism, we can deduce

$$\tilde{\phi}_+(e^{ih}z^n) = e^{i\tilde{\phi}_+(h)}(\tilde{\phi}_+(z))^n$$

It is easily shown that $1 - w^*w \in I_+$.

Similarly, using ϕ^-, we get $1 - ww^* \in I_-$ also. Hence

$$(1 - w^*w)\mathcal{M}(B \otimes \mathcal{K})(1 - ww^*) = 0$$

Consequently, w is an extremal partial isometry.

Corollary 5.3. *If ϕ is a *-homorphism from $C(\mathbb{T})$ to B a C^*-algebra, then it is extremal preserving.*

proof. It is straightforward.

By applying the definition4.1 to S and z we have $\mathcal{E}(S, B)$ be the set of the pairs (ϕ_+, ϕ_-) s.t.

(1) $\phi_+ : S \to \mathcal{M}(B \otimes \mathcal{K})$ *-homomorphism, $\phi_- : S \to \mathcal{M}(B \otimes \mathcal{K})$ completely positive contractive s.t.
  (a) $\tilde{\phi}_-(z)$ is extremal partial isometry for the unitary element $z \in \tilde{S} = C(\mathbb{T})$

  (b) $\tilde{\phi}_-(z^n) = (\tilde{\phi}_-(z))^n$ $\tilde{\phi}_-((z^*)^n) = (\tilde{\phi}_-(z^*))^n$ for the unitary element $u \in \tilde{S}$
(2) $\tilde{\phi}_+(z) - \tilde{\phi}_-(z) \in B \otimes \mathcal{K}$ thus $(\tilde{\phi}_+(z))^*\tilde{\phi}_-(z) \in 1 + B \otimes \mathcal{K}$

We call such a pair (ϕ_+, ϕ_-) as *an extremal cycle* and denote the set of extremal cycles by $\mathcal{E}(S, B)$. Two extremal cycles $(\phi_+, \phi_-), (\psi_+, \psi_-)$ are *homotopic* when there is an extremal cycle from S to $\mathcal{M}([0, 1]B \otimes \mathcal{K})$ i.e. $(\lambda_+^t, \lambda_-^t) \in \mathcal{E}(S, B), t \in [0, 1]$ s.t.

(1) the maps $t \to \lambda_+^t(f) + 1$ and $t \to \lambda_-^t(f) + 1$ from $[0, 1]$ to $\mathcal{M}(B \otimes \mathcal{K})$ are strictly continuous.
(2) For each t, $(\lambda_+^t(f) + 1)^*, \lambda_-^t(f) + 1$ are composable and the map $t \to (\lambda_+^t(f) + 1)^*\lambda_-^t(f) + 1$ from $[0, 1]$ to $1 + B \otimes \mathcal{K}$ is norm continuous.
(3) $(\lambda_+^0, \lambda_-^0) = (\phi_+, \phi_-), \quad (\lambda_+^1, \lambda_-^1) = (\psi_+, \psi_-)$

We write $(\phi_+, \phi_-) \sim (\psi_+, \psi_-)$ in this case.

Remark 5.4. *In fact, since $\mathcal{E}(S) = \mathcal{E}(\tilde{S}) = \mathcal{E}(C(\mathbb{T})) = \mathcal{U}(C(\mathbb{T}))$, the set of homotopy classes of $\mathcal{E}(S)$ is $K_1(C(\mathbb{T})) = \{[z(t)^n]\} \cong \mathbb{Z}$. By*

proposition5.1 and corollary3.4, for (ϕ_+, ϕ_+) to be extremal cycle, it is enough to consider $f(t) \in S$ (or $z(t) \in C(\mathbb{T})$).

Definition 5.5. *We let $E_\infty(S, B) \stackrel{def}{=} \mathcal{E}(S, B)/\sim$ denote the homotopy classes of extremal cycles.*

Our starting point is the following proposition which tells us an intimate relationship between KK-theory and K-theory.

Lemma 5.6. *Let w be a extremal partial isometry in $\mathcal{M}(B \otimes \mathcal{K})$. Then there is a strictly continuous path $w_t, t \in [0, 1]$, of extremal partial isometries such that $w_0 = 1, w_1 = w$.Furthermore, if w is of the form $1 + B$, then we can take $w(t)$ of the form $1 + B \otimes \mathcal{K}$.*

proof Since $B \otimes \mathcal{K}$ is stable C^*-algebra, there is a path v_t , $t \in]0, 1]$, of isometries in $\mathcal{M}(B \otimes \mathcal{K})$ such that

 (1) the map $t \mapsto v_t$ is strictly continuous,
 (2) $v_1 = 1$, and
 (3) $\lim_{t \to 0} v_t v_t^* = 0$ in the strict topology

See [JenThom]. Set $w(t) = v_t w v_t^* + 1 - v_t v_t^*$, $t \in]0, 1]$, and $w_0 = 1$. We leave the reader to check w_t has the desired properties.

Proposition 5.7. *$KK(S, B) \cong K_1(B)$ where S is suspension and B is trivially graded stable C^*-algebra.*

proof. We give a proof based on Cuntz picture of KK. Recall that

$$KK(S, B) = \{[\phi_+, \phi_-] | \phi_+, \phi_- : S \to \mathcal{M}(B \otimes \mathcal{K}) \text{ s.t. } \phi_+ - \phi_- \in B \otimes \mathcal{K}\}$$

For this definition, you can refer p155-156 [Bl].
Observe that any *-homomorphism ϕ from S into a unital C^*-algebra defines a unitary $\phi(f)+1$ where $f(t) = e^{2\pi it}-1$. Conversely, any unitary u defines a homomorphism by sending f to $u-1$. Two homomorphisms are homotopic if and only if the corresponding unitaries are homotopic. From this,if we let U_+, U_- be the unitaries which come from ϕ_+, ϕ_- respectively then we get

$$KK(S, B) \cong \{[U_+, U_-] | U_+ - U_- \in B \otimes \mathcal{K} \ U_+, U_- \in \mathcal{M}(B \otimes \mathcal{K})\}$$

Note that $U_+^* U_-$ is a unitary in $(B \otimes \mathcal{K})^\sim$ which is a unitization of $B \otimes \mathcal{K}$. So we can define a map $\Delta : KK(S, B) \to K_1(B \otimes \mathcal{K})$ by $\Delta([U_+, U_-]) = [U_+^* U_-]_1$. Since $[U_+, U_-] = [V_+, V_-]$ implies that there exist maps $t \to W_\pm^t$ from $[0, 1]$ to $\mathcal{M}(B \otimes \mathcal{K})$ s.t. $t \to W_+^t - W_-^t$ is continuous in norm $B \otimes \mathcal{K}$ and $(W_+^0, W_-^0) = (U_+, U_-)$, $(W_+^1, W_-^1) = (V_+, V_-)$, $U_+^* U_-$ is homotopic to $V_+^* V_-$. Hence Δ is well defined. Also, you can easily check $[U_+, U_-]$ are degenerate if and only if $U_+ = U_-$. To prove Δ is surjective, let U be the unitary in $(B \otimes \mathcal{K})^\sim$. As we

oberved, there is a unique map $\phi : S \mapsto B \otimes \mathcal{K}$ s.t. $\phi(z - 1) = U - 1$. Then $\Delta([0, \phi]) = U$. We remark that we could associate to $[0, \phi]$ Kasparov module $[(\hat{\mathbb{H}}_B, \phi, 0)]$ using compact perturbation.

Now let $\Delta([U_+, U_-]) = \Delta([V_+, V_-])$. Then there is a homotopy between $U_+^* U_-$ and $V_+^* V_-$. let $P(t)$ be the corresponding homotopy in $1 + (B \otimes \mathcal{K})$. Since there are strictly continuous maps $V(t) : [0, 1] \mapsto \mathcal{M}(B \otimes \mathcal{K})$, $W(t) : [0, 1] \mapsto \mathcal{M}(B \otimes \mathcal{K})$ which connects U_+ and V_+, U_- and V_- respectively by lemma 5.6, then $(W(t)P(t)^*, V(t)P(t)$ is a homotopy between (U_+, U_-) and (V_+, V_-). Hence we have proven Δ is injective.

The following lemmas are the key facts in this paper which employ nice properties of S and $\mathcal{M}(B \otimes \mathcal{K})$.

Lemma 5.8. *Given an extremal partial isometry v in a unital C^*-algebra B, there is a completely positive contractive map from S to B which sends f to $v - 1$. In fact, there is the unique completely positive unital map $\tilde{\phi}$ from $C(\mathbb{T})$ to B which sends z to v such that $\tilde{\phi}(z^n) = v^n$ and $\tilde{\phi}(z^{-n}) = (v^*)^n$ for $n \geq 0$.*

proof. By lemma2.4, it is enough to show that there is a contractive positive map $\phi : S \to B$ which send f to $v - 1$. Let I_+, I_- be defect ideals of v. Define $\tilde{\phi}$ by $\tilde{\phi}(p(e^{i\theta}) + \overline{q(e^{i\theta})}) = p(v) + q(v)^*$ where p, q are polynomials in $C(\mathbb{T})$. In B/I_+, $\pi^+(v) = \overline{v}$ is an isometry. Therefore $\tilde{\phi}^+(e^{i\theta}) = \overline{v}$. If $\tau(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta}$ is a positive function in $C(\mathbb{T})$, then there is a function $f(z) = \sum_{n=0}^{N} b_n z^n$ such that $\tau(e^{i\theta}) = |f(e^{i\theta})|^2$. It is easy to check $\tilde{\phi}^+(f(e^{i\theta})\overline{f(e^{i\theta})}) = f(\overline{v})f(\overline{v})^*$ since $\overline{v}$ is isometry. Hence, $\tilde{\phi}^+$ is positive. By Russo-Dye theorem, $\|\tilde{\phi}^+\| = \|\tilde{\phi}^+(1)\| = 1$. By this, $\tilde{\phi}^+$ is also contractive. Similarly, we can show that $\tilde{\phi}^-$ is positive and contractive. Then, by the proposition2.8, $\tilde{\phi}$ is completely positive and contractive. Let ϕ be the restriction of $\tilde{\phi}$ to S. Then ϕ is also completely positive and

$$\sup_{a \in S} \frac{\|\phi(a)\|}{\|a\|} = \sup_{a \in S} \frac{\|\tilde{\phi}(a)\|}{\|a\|} \leq \sup_{a \in C(\mathbb{T})} \frac{\|\tilde{\phi}(a)\|}{\|a\|} \leq 1$$

implies it is also contractive. From the definition of $\tilde{\phi}$, it is unique.

Lemma 5.9. *The unitary group of $\mathcal{M}(B \otimes \mathcal{K})$ is path-connected in norm topology.*

proof. This is well-known Kuiper-Mingo's theorem. The proof of this theorem can be found in [CH] or [Mi].

Now we are ready to prove the main result of this paper. By our observation, if we let u, v be an unitary element in $\mathcal{M}(B \otimes \mathcal{K})$, an extremal partial isometry in $\mathcal{M}(B \otimes \mathcal{K})$ from $\tilde{\phi}_+$, $\tilde{\phi}_-$ respectively and

denote by $[u, v]$ its homotopy class, then we have natural map which send $[\phi_+, \phi_-]$ to $[u, v]$. In fact, we have

$$E_\infty(S, B) \cong \{[u, v] \mid u^*v \in \mathcal{E}(\mathcal{M}(B \otimes \mathcal{K})) \& u^*v \in 1 + B \otimes \mathcal{K}\}$$

Clearly, this map is well-defined and it is surjective by the lemma5.8. Suppose $\tilde{\phi_+^i}(z) = u_i$, $\tilde{\phi_-^i}(z) = v_i$ for $i = 0, 1$ If $[u_0, v_0] = [u_1, v_1]$, there is a homotopy between (u_0, v_0) and (u_1, v_1). i.e. there are strictly continous maps $\lambda_\pm : [0, 1] \mapsto U(\mathcal{M}(B \otimes \mathcal{K}))$ s.t. $\lambda_+ - \lambda_-$ is norm-continuous map $\&$ $(\lambda_+(i), \lambda_-(i)) = (u_i, v_i)$ for $i = 0, 1$. By lemma 5.8, there are corresponding maps $\lambda_\pm^t : S \mapsto \mathcal{M}(B \otimes \mathcal{K})$ for each $t \in [0, 1]$. We must show the strict continuity of each map. Using the fact $\{\sum_{n=-k}^{n=k} a_n z^n \mid a_n \in \mathbb{C}\}$ is dense in $C(\mathbb{T})$, it's enough to show $t \to \lambda_\pm^t(\sum_{n=-k}^{n=k} a_n z^n)T$ where $T \in \mathcal{M}(B \otimes \mathcal{K})$ is norm continuous with respect to t. Given $\epsilon > 0$, $\|\lambda_\pm^t(\sum_{n=-k}^{n=k} a_n z^n)T - \lambda_\pm^s(\sum_{n=-k}^{n=k} a_n z^n)T\| \leq \|\sum_{n=-k}^{n=k} a_n(\lambda_\pm^t(z)^n - \lambda_\pm^s(z)^n)T\|$. Now let δ be such that if $|t - s| < \delta$ then $\|(\lambda_\pm(t))^n - (\lambda_\pm(t))^n T\| < \frac{\epsilon}{2k+1 sup_n|a_n|}$ for each $n = -k, -k + 1, \ldots, k - 1, k$. Therefore if $|t - s| < \delta$ we have $\|\sum_{n=-k}^{n=k} a_n(\lambda_\pm^t(z)^n - \lambda_\pm^s(z)^n)T\| < \sum_{n=-k}^{n=k} |a_n|\|(\lambda_\pm(t))^n - (\lambda_\pm(t))^n T\| < \epsilon$.
Similarly, $t \to T\lambda_\pm^t(\sum_{n=-k}^{n=k} a_n z^n)$ is shown to be norm-continuous with respect to t and $t \to \lambda_+^t(f) - \lambda_-^t(f)$ is shown to be norm-continuous with respect to t for each $f = \sum_{n=-k}^{n=k} a_n z^n$.
Finally we should check that $(\lambda_+^i, \lambda_-^i) = (\phi_+^i, \phi_-^i)$ for $i = 0, 1$. But both maps $\tilde{\lambda_\pm^i}$ and $\tilde{\phi_\pm^i}$ send z to same extremal partial isometries u_i and v_i. In addition, both maps send z^n to u_i^n , v_i^n and send $(z^*)^n$ to $(u_i^*)^n, (v_i^*)^n$. From the uniqueness of lemma 5.8, the conclusion follows.

Since $K_1(B)$ is embedded in $[\mathcal{E}_\infty(B)]$, we can think of a set containing $KK(S, B)$ which extends the map $\Delta : KK(S, B) \to K_1(B)$. As indicated above (proposition4.3), we have the following theorem.

Theorem 5.10. *There is a bijection $\Delta_e : [\mathcal{E}_\infty(S, B)] \to [\mathcal{E}_\infty(B)]$ such that the following diagram is commutative.*

$$\begin{array}{ccc} [\mathcal{E}_\infty(S, B)] & \xrightarrow{\Delta_e} & [\mathcal{E}_\infty(B)] \\ \uparrow & & \uparrow \\ KK(S, B) & \xrightarrow{\Delta} & K_1(B) \end{array}$$

proof. We define the map Δ_e by $\Delta_e([u, v]) = u^*v$ as we have defined Δ.

It's not hard to check well-definess of the map.(It is almost same as the well-definess of the map Δ.)

Surjectivity: Let v be the extremal partial isometry in $\widetilde{B \otimes \mathcal{K}}$. Since the quotient map from $\widetilde{B \otimes \mathcal{K}}$ onto $\frac{\widetilde{B \otimes \mathcal{K}}}{B \otimes \mathcal{K}} = \mathbb{C}$ is extremal preserving map, the scalar part of v is also the extremal partial isometry in $\mathbb{C}$. If v is written as $\lambda + T$ where $T \in B \otimes \mathcal{K}$ and $\lambda \in \mathbb{C}$, $(1 - |\lambda|^2)\mathbb{C}(1 - |\lambda|^2) = 0$. Hence $1 - |\lambda|^2 = 0$. In other words, $\lambda \in \mathbb{T}$. Hence we may assume v is of the form $1 + B \otimes \mathcal{K}$ if necessary to multiply $\bar{\lambda}$. Then there is a map $\phi : S \to \mathcal{M}(B \otimes \mathcal{K})$ such that $\phi(f) + 1 = v$ by the lemma 5.8. Then $\Delta_e([0, \phi]) = [v]$.

Injectivity: Let $\Delta_e([u_0, v_0]) = \Delta_e([u_1, v_1])$. We may assume $(u_0)^* v_0 = (u_1)^* v_1$. Let $\lambda_+ : [0, 1] \mapsto U(\mathcal{M}(B \otimes \mathcal{K}))$ be a norm-continuous map between u_0 and u_1 by lemma 5.9. Set $\lambda_-(t) = \lambda_+(t)((u_0)^* v_0)$. Then

$$\lambda_-(0) = \lambda_+(0)(u_0)^* v_0 = v_0$$

$$\lambda_-(1) = \lambda_+(1)(u_1)^* v_1 = v_1$$

Also, $\lambda_+(t) - \lambda_-(t) = \lambda_+(t)(1 - (u_0)^* v_0) \in B \otimes \mathcal{K}$ for all $t \in [0, 1]$. Therefore (λ_+, λ_-) is a homotopy between (u_0, v_0) and (u_1, v_1) as we wanted.

Theorem 5.11. *There is a bijective map Δ_k from $KK_e(S, B)$ onto $K_e(B)$ such that the following diagram is commutative.*

$$
\begin{array}{ccc}
KK_e(S, B) & \xrightarrow{\ \Delta_k\ } & K_e(B) \\
\kappa_e \uparrow & & \uparrow \kappa \\
[\mathcal{E}_\infty(S, B)] & \xrightarrow{\ \Delta_e\ } & [E_\infty(B)]
\end{array}
$$

proof. For each extremal cycle (ϕ_+, ϕ_-), we shall denote the element of $KK(S, B)$ by $[(\phi_+, \phi_-)]$ (or equivalently, $[u, v]$) To avoid confusion we shall denote by $[(\phi_+, \phi_-)]_\infty$ its homotopy class in $\mathcal{E}_\infty(S, B)$. Similarly, for each w in $\mathcal{E}(\widetilde{B \otimes \mathcal{K}})$, we shall denote by $[w]$ its equivalent class in $K_e(\widetilde{B \otimes \mathcal{K}})$ and by $[w]_\infty$ its homotopy class in $E_\infty(\widetilde{B \otimes \mathcal{K}})$. Now define $\Delta_k([\phi_+, \phi_-])$ by $[u^* v]$ where $u = \tilde{\phi}_+(z)$ and $v = \tilde{\phi}_-(z)$. Let $[(\phi_+, \phi_-)] = [(\psi_+, \psi_-)]$. Then there is (τ_+, τ_-) such that $\tilde{\tau}_-(z) = v$ has smaller defects ideals than $\tilde{\phi}_-(z) = v_0$ and $\tilde{\psi}_-(z) = v_1$ s.t.

$$\left(\Theta_B \circ \begin{bmatrix} \phi_+ & 0 \\ 0 & \tau_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \phi_- & 0 \\ 0 & \tau_- \end{bmatrix} \right) \sim \left(\Theta_B \circ \begin{bmatrix} \psi_+ & 0 \\ 0 & \tau_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \psi_- & 0 \\ 0 & \tau_- \end{bmatrix} \right)$$

Since Θ_B is isomorphism, this implies

$$\left(\begin{bmatrix} \tilde{\phi}_+(z) & 0 \\ 0 & \tilde{\tau}_+(z) \end{bmatrix}\right)^* \begin{bmatrix} \tilde{\phi}_-(z) & 0 \\ 0 & \tilde{\tau}_-(z) \end{bmatrix} \sim \left(\begin{bmatrix} \tilde{\psi}_+(z) & 0 \\ 0 & \tilde{\tau}_+(z) \end{bmatrix}\right)^* \begin{bmatrix} \tilde{\psi}_-(z) & 0 \\ 0 & \tilde{\tau}_-(z) \end{bmatrix}$$

Therefore $[u_0^* v_0]_\infty + [u^* v]_\infty = [u_1^* v_1]_\infty + [u^* v]_\infty$ i.e. $[u_0^* v_0]_\infty \approx [u_1^* v_1]_\infty$
So far we have shown Δ_k is well-defined.
From the definition of Δ_k, the commutativity of diagram follows easily so that the map is surjective.
It remains only to show the map is injective. For this let $[u_0^* v_0] = [u_1^* v_1]$ in $K_e(\widetilde{B \otimes \mathcal{K}})$. Note that defect ideals of $u_0^* v_0$ and $u_1^* v_1$ are same to defect ideas of v_0 and v_1. Therefore there is v in $\mathcal{E}(\widetilde{B \otimes \mathcal{K}})$ such that v has smaller defect ideals that v_0 and v_1 s.t. $u_0^* v_0 v \sim u_1^* v_1 v$. In other words, $[(u_0, v_0)]_\infty + [(1, v)]_\infty = [(u_1, v_1)]_\infty + [(1, v))]_\infty$. This implies the map is injective by the routine argument.

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